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Spinor Waves in a Space-Time Lattice (II)

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"Dreifach ist der Schritt der Zeit:..."
Schiller: Sprüche des Konfuzius

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In a previous note, an exceptional space-time lattice was found by a roundabout heuristic process. This process was far from convincing; here a more translucent characterization of the lattice is presented. A cornerstone is the consideration of pairs of reciprocal lattices, together with the basic symmetry (S_4) of the metric tensor. The basic requirement is that one member of a pair of reciprocal lattices contains the other as a sublattice. One preferred lattice is discussed in some detail; it contains three copies of its reciprocal lattice, and it is the simplest example satisfying the requirements. In the expression of the metric tensor in terms of the lattice generators a possible topology on the lattice is suggested. By means of this topology, propagation of spinor waves can be formulated. This proposed—the simplest—propagation mechanism is inhibited, though, by the fact that the three sublattices are required to carry the two types of spinors alternatively. This inhibition can be lifted by introducing a second type of elementary propagation, to next nearest neighbors. If this inhibition is only feebly lifted, this would result in particles with mass small as compared to the inverse of the lattice constant, presumably the Planck mass. Including the propagation to next nearest neighbors leads to spinor waves with six components, two components for each sublattice. In the long-wavelength limit four of them obey a massive Dirac equation, while the remaining two obey a Weyl equation. These considerations conceivably provide a root for the lack of parity invariance in nature, and for the joint occurrence of pairs of massive and massless spinor waves. The construction, furthermore, allows one to accommodate just three different families of spinor waves of this type. Extension of the above arguments outside the realm of the long-wavelength limit forcibly makes the lattice concept independent of the original continuous Minkowski space-time: the latter is no longer a unique embedding space for the lattice, but appears as an *approximate interpolation, valid near the long-wavelength limit.* This may be the minimal requirement to be imposed on a lattice theory in the light of the empirical evidence, if the scale of the lattice structure is, compared to the empirical scales, as small as the Planck scale.

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1. A SPACE-TIME LATTICE: SKETCH OF AN ARGUMENTATION

Consider in a four-dimensional affine space a set of four contravariant vectors b_M^κ and its reciprocal set of four covariant vectors a_λ^N. The κ, λ denote the space-time indices, the M, N distinguish the four vectors. The reciprocity is expressed by

$$a_\kappa^N b_M^\kappa = \delta_M^N$$

Consider the contra- and covariant lattices

$$\{L(b)\}: \quad \mu^M b_N^\kappa$$

$$\{L(a)\}: \quad \nu_N b_\lambda^N$$

where μ^M and ν_N are $\equiv 0 \bmod 1$.

The b and a can be taken as bases of the coordinate systems for the contra- and covariant vector spaces, respectively:

$$b_M^\kappa = \delta_M^\kappa, \qquad a_\lambda^N = \delta_\lambda^N$$

The lattices $L(a)$ and $L(b)$ can be compared after introduction of a metric tensor, which thus finds its fundamental role in the framework of a lattice theory:

$$b_{M\kappa} = g_{\kappa\lambda} b_M^\lambda = g_{\kappa M}$$

At this point a reasonable symmetry requirement for the metric tensor is introduced. As is well known, quadratic forms that are permutation symmetric in the coordinates will be either definite or have a signature with one deviating sign, so that a permutation-symmetric form of the metric tensor is possible in Minkowski space-time. It is assumed that the coordinates defined by the pair of reciprocal lattices $L(a)$ and $L(b)$ are such that the metric tensor assumes this permutation-symmetric form. This assumption seems natural, as the four generators of the lattices will play equivalent roles. A permutation-symmetric $g_{\kappa\lambda}$ depends on only two parameters:

$$g_{\kappa\lambda} = \begin{pmatrix} \xi & \eta & \eta & \eta \\ \eta & \xi & \eta & \eta \\ \eta & \eta & \xi & \eta \\ \eta & \eta & \eta & \xi \end{pmatrix}$$

The parameters ξ and η are restricted only by the requirement that the metric be Minkowskian rather than definite. This condition reads

$$(\xi + 3\eta)(\xi - \eta)^3 < 0$$

that is, for positive, η,

$$\eta > -\ \xi/3 \qquad \text{and} \qquad \eta > \xi$$

(correspondingly for negative η).

The smallest pointset, for the combination of $L(a)$ and $L(b)$, is obtained by the requirement that one of the pair of lattices be a sublattice of the other. In view of the equivalence of the roles played by the two lattices, it is no restriction to assume that $L(b)$ be a sublattice of $L(a)$.

Expressing now $b_{M\kappa}$ in terms of the a_κ^N by means of the foregoing explicit formulas yields

$$b_{1\kappa} = \xi a_\kappa{}^1 + \eta(a_\kappa{}^2 + a_\kappa{}^3 + a_\kappa{}^4) \quad \text{cycl}$$

$L(b)$ is a sublattice of $L(a)$ only if ξ and η are integers.

In general $L(a)$ contains a number of congruent copies of $L(b)$; that multiplicity, B, is given by the determinant of the four vectors $b_{\kappa M}$, the volume of the unit cell of the b-lattice in terms of that of the a-lattice:

$$B = (3\eta + \xi)(\eta - \xi)^3$$

The lowest multiplicity compatible with the conditions on ξ and η is three, the value obtained for $\xi = 0$, $\eta = 1$.[2] The a-lattice then has three consecutive sublattices $L(b;1)$, $L(b;2)$, $L(b;3)$, which are arranged in cyclic order as will be explained below.

Some properties of this exceptional lattice are the following:

(i) $b_M \cdot b_N = 1 - \delta_{MN}$, $a^M \cdot a^N = 1/3 - \delta^{MN}$.

(ii) $a^M \cdot a^0 - 1/3\ \forall M = 1, \ldots, 4$, where $a^0 = -\sum_1^4 a^M$.

(iii) $b_1 = a^2 + a^3 + a^4$ cycl, or $b_1 = -a^0 - a^1$.

(iv) $b_M - b_N = a^N - a^M$, $a^N - a^M \equiv 0 \bmod b$, $M, N = 1, \ldots, 4$.

(v) $-a^0 - a^N \equiv 0 \bmod b$, $N = 1, \ldots, 4$.

(vi) Let P be a point on $L(b; j)$, one of the sublattices; then translating P by $-a^N$ to Q^N yields always a point on $L(b; j+1)$, for the four translations $-a^N$ with $N = 1, \ldots, 4$, as well as for the translation a^0.

(vii) $a^L + a^M + a^N \equiv 0 \bmod b$.

(viii) Finally, another identity is to be noted: the specific metric tensor ($\xi = 0$, $\eta = 1$) can be expressed in terms of the a_κ^M by

$$g_{\kappa\lambda} = \left(\sum_M a_\kappa^M\right)\left(\sum_N a_\lambda^N\right) - \sum_M a_\kappa^M a_\lambda^M = a_\kappa^0 a_\lambda^0 - \sum_M a_\kappa^M a_\lambda^M$$

or, with the abbreviations

$$G_{00} = 1, \qquad G_{MM} = -1, \qquad \text{all other } G_{MN} = 0$$

[2]Finkelstein and Gibbs (1993) reach the same form for the line element along somewhat different lines.

by

$$g_{\kappa\lambda} = G_{MN} a^M_\kappa a^N_\lambda \qquad \text{summed over } M, N = 0, 1, \ldots, 4$$

which is the initial assumption in Wouthuysen (1993) [hereafter referred to as (I)].

The property (viii) suggests a topology in the a-lattice consistent with the symmetry group S_4, the symmetry of the four generators of the lattice. The five points obtained from an arbitrary point P in $L(a)$ by translation over $-a^M$, a^0 are defined to be the neighbors of P. Note the one-sidedness of this definition: all five neighbors of $P \subset L(b; j)$ are situated on $L(b; j+1)$, where $j+1$ is taken modulo 3. They are the only points with that property that share a unit cell with P.

A more direct motivation for the remarkable composite lattice of (I) has thus been sketched. The motivation is *not* a derivation, evidently, but it puts in relief a basis for the objective existence, in Minkowski space-time, of a class of exceptional lattices.

2. FURTHER PROPERTIES

The a-lattice has net-spaces of three dimensions which, viewed in an appropriate coordinate frame, are spaces of constant time. The three-dimensional lattice in such a cross section is a cubic closest packing. The lattice $L(a)$ can most conveniently be visualized as the succession in time of these cross-section spaces. In consecutive cross sections the three-lattices are translated with respect to each other by a standard three-vector. The situation can be visualized by remarking that the cubic closest packing can be considered as the configuration of the vertices of a set of identical—in size and orientation—tetrahedrons, four of which meet in every vertex. The centers of these tetrahedrons again are arranged in a cubic closest packing configuration, and they form the next three-dimensional cross section of $L(a)$. After four steps the original configuration is recovered: the pattern repeats itself with period four.

Numbering these net-space configurations by $c_1, \ldots, c_4$, the lattice $L(a)$ is built according to the scheme

$$L(a): \quad c_1\, c_2\, c_3\, c_4\, c_1\, c_2\, c_3\, c_4\, c_1\, c_2\, c_3\, c_4 \cdots$$

whereas

$$L(b; 1): \; c_1 \qquad c_4 \qquad c_3 \qquad c_2 \qquad c_1 \cdots$$

and

$$L(b; 2): \quad c_2 \qquad c_1 \qquad c_4 \qquad c_3 \qquad c_2 \cdots$$

etc.

The configuration c_{j+1} arises from c_j by contracting each one of its tetrahedrons to its respective center; c_j is obtained from c_{j+1} by the reverse construction.

3. FIELDS AND FIELD EQUATIONS

The exceptional lattice admits as its local invariance group the permutation group S_4, that is, the group of permutations of four objects, in every point of $L(a)$.

This group has, among others, a two-dimensional complex representation. It can be visualized as the spinor representation of the rotations that leave a cube invariant. A two-component spinor is therefore a possible "geometric object" for the local group. This, together with the empirical fact of the prominence of spinor fields in physics of elementary particles—leptons, quarks—is taken as a motivation to study spinor fields on the exceptional lattice.

The S_4 spinors allow generalization to relativistic spinors in two ways, to the two types of relativistic spinors, representations (1/2, 0) and (0, 1/2) of $SL(2C)$, in what follows denoted by u-spinors and v-spinors, respectively. Correspondingly, there are two matrix four-vectors:

$$\sigma^{\kappa} = (1, \sigma) \qquad \text{operating on } u\text{-spinors}$$

$$\sigma'^{\kappa} = (-1, \sigma) \qquad \text{operating on } v\text{-spinors}$$

(σ are the three Pauli matrices).

Field equations are formulated by defining the elementary propagation from one sublattice to the next. The elementary propagation from a point P to one of its neighbors Q^M will involve the connecting four-vector a^M ($M = 0, 1, \ldots, 4$); an obvious proposal for the contribution to the spinor $v(Q^M)$ in Q^M by the spinor $u(P)$ in P is

$$\delta v = a^M_{\kappa} \sigma^{\kappa} u(p)$$

The v-spinor in Q is accordingly defined as the sum of the contributions from the five points P which have Q as their common neighbor:

$$v(Q) = \sum_m (a^M \cdot \sigma) U(Q - a_M)$$

The alternation of spinor types on successive sublattices, as suggested by this prescription of elementary propagation, *cannot be carried out for the three consecutive sublattices*. A choice has to be made. Suppose the sublattice $L(b; 1)$—in abbreviated notation L_1—to be the seat of a u-spinor field U, L_2 carries a v-spinor v, and L_3 again a u-spinor, denoted by u to distinguish it from the u-spinor U on L_1.

The following field equations then suggest themselves, in a transparent notation:

$$v(Q) = \sum_K (a^K \cdot \sigma) U(Q - K)$$

$$u(R) = \sum_K (a^K \cdot \sigma') v(R - K)$$

$$0 = \sum_K (a^K \cdot \sigma) u(P - K)$$

These equations of motion respect global Lorentz invariance; but they seem to contradict the usual concepts of causality because influences spread in spacelike directions for $K = 1, \ldots, 4$. *Nevertheless, wave solutions of these equations, in the long-wavelength limit, will be shown to obey propagation (differential) equations of the usual causal type, at least in an extremely good approximation.*

Indeed, by substituting plane waves proportional to $\exp(ip \cdot x)$ into the equations of motion, with amplitudes u_0, v_0, U_0, respectively, one obtains

$$v_0 = \sum (a^K \cdot \sigma) \exp(-ip \cdot a_K) U_0$$

$$u_0 = \sum (a^K \cdot \sigma') \exp(-ip \cdot a_K) v_0$$

$$0 = \sum (a^K \cdot \sigma) \exp(-ip \cdot a_K) u_0$$

In the limit of long wavelength, i.e., $p_\lambda \ll 1$, and in view of

$$\sum a^K = 0 \qquad \text{(sum over all five values of } K\text{)}$$

this leads to

$$v_0 = -ip \cdot \sigma U_0 \tag{1}$$

$$u_0 = -ip \cdot \sigma' v_0 \tag{2}$$

$$0 = -ip \cdot \sigma u_0 \tag{3}$$

Substitute (2) into (3):

$$(p \cdot p) v_0 = 0$$

(1) into (2):

$$(p \cdot p) U_0 = u_0$$

The determinant of (3) being $(p \cdot p)$, if $(p \cdot p) \neq 0$, then $u_0 = 0$, entailing $U_0 = 0$, $v_0 = 0$. So that necessarily $(p \cdot p) = 0$, leading to $u_0 = 0$; (1) and (2) then yield

$$v_0 = ip \cdot \sigma U_0$$
$$0 = -ip \cdot \sigma' v_0$$

In the long-wavelength limit, therefore, the solutions are degenerate, one of the three spinor degrees of freedom being quenched. The equations of motion can be simplified, leaving out (3). The dispersion law is $(p \cdot p) = 0$. The plane wave solutions in this limit are solutions of a "skew" Dirac equation of the type

$$i\gamma^{\kappa}\partial_{\kappa}\psi = 1/2(1+\gamma^5)\psi$$

The equations that remain have a "hierarchical" structure: the spinor field U plays a key role in the sense that if it is zero, all fields are zero.

This feature will be maintained in the following generalization.

The missing degrees of freedom can be called to life by disturbing the equations of motion. Dislocations of the lattice could be invoked, as suggested in (I), as a topological perturbation achieving this aim. This idea proved after all unsatisfactory. Nevertheless, the situation is explored by formally perturbing the system by a small term, so as to embed the degenerate case, as follows:

$$v(Q) = \sum_K (a^K \cdot \sigma) U(Q - K)$$

$$u(R) = \sum_K (a^K \cdot \sigma') v(R - K)$$

$$\varepsilon v(Q) = \sum_K (a^K \cdot \sigma) u(Q - 2K)$$

Thus a direct transmission is described from a special set of next nearest neighbors: the block uU is effectively broken in this way. The transmission occurs in the same directions as usual, but with a double steplength, while the effect on the v_Q is reduced by a factor ε [the notation $u(Q - 2K)$ stands for the u-spinor in the point $-2a_K$ removed from Q].

These equations again lead to conditions for the amplitudes of a plane wave solution in the limit of long wavelength:

$$v_0 = -i(p \cdot \sigma) U_0$$
$$u_0 = -i(p \cdot \sigma') v_0$$
$$\varepsilon v_0 = -2i(p \cdot \sigma) u_0$$

Therefore p has to satisfy the equation

$$(p \cdot p)(p \cdot p - \varepsilon/2)^2 = 0$$

If $p \cdot p = 0$, u_0 and v_0 are both $=0$, and U satisfies a Weyl equation. On the other hand, if $p \cdot p = \varepsilon/2$ for positive values of ε, u and v obey a Dirac equation, with the mass m satisfying $m^2 = \varepsilon/2$, while

$$U = (2/\varepsilon)u$$

Slightly redefining the variables as

$$\varepsilon/2 = m^2, \qquad U' = m^2 U, \qquad v' = mv$$

one obtains

$$\sum (a^K \cdot \sigma)u = mv'$$

$$\sum (a^K \cdot \sigma')v' = mu$$

$$\sum (a^K \cdot \sigma)U' = mv'$$

The key role of U' is again apparent: $U' = 0$ entails $u = v' = 0$.

The dispersion law for long wavelengths reads

$$(p \cdot p)(p \cdot p - m^2) = 0$$

The degrees of freedom corresponding to $(p \cdot p) = 0$ have $u = v' = 0$, U' satisfies

$$(p \cdot \sigma)U' = 0, \qquad \text{i.e., a Weyl equation}$$

The degrees of freedom corresponding to $(p \cdot p) = m^2$ have $U' = u$ and u, v' obey a Dirac equation with mass m.

The formal introduction of the ε term into the equations of motion, describing a transmission, across the uU block, to a special set of next nearest neighbors, therefore leads to a desirable perspective: the joint appearance, in the long-wavelength limit, of a two-component field (Weyl field) and a massive Dirac field, a situation characteristic for the leptonic degrees of freedom. It therefore is of interest to try and find a possible origin of the ε term. The following option presents itself. The ε term describes the influence on the v-spinor of the twice removed u-spinor, via the interposed U-spinor. It seems not unnatural to assume an effect, on the transmission from u toward v, of the excitation of the interposed U field, specifically in its independent mode. With the notation U_ν for this Weyl field, the proposed term would read

$$C(U_\nu{}^+(Q-K)U_\nu(Q-K))v(Q)$$

and the relevant equation in full would be

$$C(U_\nu{}^+(Q-K)U_\nu(Q-K))v(Q)=\sum_K (2a^K\cdot\sigma)u(Q-2K)$$

In this way one can achieve, in the presence of a "neutrino background," an effectively constant coefficient leading to a mass term for the lepton, caused by the omnipresent neutrino background. Note that hereby is introduced an effectively nonlinear term in the transmission. A broader discussion of possible relations with interactions has to be postponed.

It should be apparent that, in this perspective, the smallness of particle (lepton) masses as compared to the Planck mass would be a direct consequence of the paucity of the neutrino background.

4. THREE FAMILIES

The six-component field carries two u-spinors and one v-spinor, distributed in a given way among the three sublattices. It allows, provided one assumes also some next nearest neighbor propagation, Dirac waves and Weyl waves, where the Dirac mass is determined by the "background" of the Weyl field.

Clearly three such fields can be formulated, corresponding to the three possible assignments of the two u-spinors and v-spinor to the three sublattices. This circumstance could be at the root of the existence of three families of leptons. In each of the three six-component fields the mass of the Dirac lepton will be determined by the corresponding Weyl background.

5. AWAY FROM THE LONG-WAVELENGTH LIMIT

The equations (1)–(3) and their subsequent generalization resulted from an approximation, keeping only the first two terms of the power series of the exponential $\exp(-ip\cdot a_K)$. As a consequence, the equations of motion in this approximation contain p only linearly (Dirac and Weyl equations). They read, without this approximation:

$$v_0=-is\cdot\sigma U_0$$

$$u_0=-is\cdot\sigma' v_0$$

$$\varepsilon v_0=-is\cdot\sigma u_0$$

where

$$-is_\kappa=\sum_{K=0}^{4} a_\kappa^K \exp(-ip\cdot a_K)$$

In the realm of long wavelengths, where the experiments are actually situated, the components of p are many powers of ten smaller than unity; the next term of the power series is therefore negligible in size, but it leads to corrections in the dispersion law which, as it stands, will be complex valued, either requiring complex values of p or lowering the dimensionality of the collection of permitted values of p. It is only along the four "cardinal null directions," the directions of the basis vectors b_K of the sublattices, that the reality of the first approximation holds throughout.

It is possible, however, to recover the usual situation, i.e., a real dispersion law, at least in the next approximation in the power series in p, *by making the a^K depend slightly on p*, in such a way that the condition

$$\sum_{KL=0}^{4} G_{KL} a_\kappa^K a_\lambda^L = g_{\kappa\lambda}$$

be maintained (at the expense of the validity of $\sum_0^4 a^K = 0$). The leading term in the power series in p then does not vanish as before, and it can be used to cancel the undesirable second-order term.

This can be achieved by submitting the a^K to a five-dimensional Lorentz transformation which leaves G_{KL} unaltered, thereby safeguarding the expression for $g_{\kappa\lambda}$ in terms of the five 4-vectors a. More explicitly, introduce a slightly altered set of five 4-vectors $a_\lambda'^K$ by performing an infinitesimal five-dimensional Lorentz transformation on the a_λ^K:

$$a_\lambda'^K = a_\lambda^K + \varepsilon_\lambda n^K, \qquad n_K = (11111)$$

where ε is an infinitesimal spacetime vector, independent of K. The zeroth-order term in $-is_\kappa$ then is $-3\varepsilon_\kappa$. The second-order term in the expression for $-is_\kappa$ reads

$$-1/2 \sum_0^4 a_\kappa^K p_\lambda p_\mu a_K^\lambda a_K^\mu$$

which added to $-3\varepsilon_\kappa$ has to yield zero. In terms of the fundamental a-lattice, the introduction of such an ε_κ amounts to a shift of one sublattice with respect to the preceding one by this four-vector. If the shift of L_2 with respect to L_1 is denoted by $\varepsilon(1)_\kappa$, and similarly for the other two sublattices, one has to require that

$$\varepsilon(1)_\kappa + \varepsilon(2)_\kappa + \varepsilon(3)_\kappa = 0$$

so as to ensure coherence of the combined lattice.

The Weyl wave found in the long-wavelength limit is a solution for which the amplitudes u_0 and v_0 are zero. The propagation equation for U between the sublattices L_1 and L_2 requires only $\varepsilon(1)$ to be different from zero so as to compensate the second-order term in the equation of motion.

The other two ε vectors are irrelevant, and no incoherence is caused in the complete lattice.

The Dirac wave involves two of the three ε vectors, at least if the a vector occurring in the next nearest neighbor influence is taken to be the one from U to v, that is, the second step of the two-step propagation. As only two of the three ε vectors then are involved, the lattice-coherence condition can again be maintained; and here again the wave can be propagated up to and including the second approximation in p. It should be evident that this is achieved for every (small) p individually: the ε vector depends on p. Thus for every (small) p an individual interpolation between lattice points leads to an individual embedding of the lattice in "the" continuum. In this way it seems possible to wean the lattice concept from an underlying continuum; the continuum that is "perceived" empirically appears only as an approximate concept, with very small lack of precision in its definition for all particle momenta on which the empirical evidence for the classical continuum is based.

6. PARTICLE ASPECT AND QUANTIZATION

The "microscopic" equations of propagation that were postulated as field equations allowed a consistent interpretation only insofar as long-wavelength wave solutions were considered. Only those aspects of the fundamental propagation correspond to waves as they are used to describe elementary particle motions. It is therefore not quite surprising that the usual (anti-) commutation rules, with their strictly local character, are not compatible with the fundamental propagation equations on the lattice. The simplest case to study the anticommutation rules is the Weyl field, where only one sublattice is involved: in absence of the u and v fields the U field obeys

$$\sum_{0}^{4} (a^K \cdot \sigma) U(Q - K) = 0$$

This equation expresses U in the point $Q - a_0$ in terms of the values in the four other points, situated in another "generation," a 3-space of constant time (in the preferred rest frame of the lattice). The evolution of anticommutation rules can be studied on the basis of this equation: evidently local equal-time anticommutators are incompatible with this evolution equation. On the other hand, it is known that in the long-wavelength limit the Weyl equation describes the solutions to a very good approximation, and the usual local equal-time anticommutators are consistent with the Weyl equation. Therefore, if the lattice description has any validity, the quantum rules play a more secondary role than is generally assumed. In the long-wavelength limit there should be no difficulties in interpreting the waves as usual. In particular, the fact that the leaking toward second nearest neighbors is proportional to the neutrino density introduces an aleatory aspect into the propagation of the Dirac waves as well, which in this way is grafted onto the quantum nature of the neutrino waves.

7. REMARKS AND PERSPECTIVES

The space-time model enunciated in (I) has here been given a fresh basis; the preliminary results may be of sufficient interest to warrant more research along these lines. The main outcome to date is the natural occurrence of three families of "leptons," if the waves studied here deserve this interpretation. Each family comprises a massive four-component field and a massless two-component ("neutrino") field. The mass is, in this scheme, induced by the density of the respective neutrinos. In the limit of mass zero and long wavelength, all families share the Weinberg–Salam symmetry group, the threefold nature of the spinor base being induced directly by the way the space-time lattice is a composite of three sublattices. The suspicion arises that the actual neutrino backgrounds of the three lepton families are in some way instrumental in stabilizing the lattice: the lattice and the three neutrino backgrounds would evolve together and would determine each other's continuation. But a detailed mechanism describing this coevolution is lacking.

Two perspectives may deserve being mentioned. First, closer examination of the dispersion law far from the long-wavelength limit shows, along the four cardinal null directions, curious bifurcations where the momentum is one third of the characteristic crystal momentum: such waves could be the basis for a description of quark degrees of freedom, as only triples of them allow an interpretation for total momentum in the empirical domain (close to zero modulo the lattice momentum).

A second aspect of the model may be of importance in cosmological considerations. Momenta along the four "cardinal" null directions have easy access to the lattice, and in the primitive universe they may be preferably populated by very energetic particles. This would be the most direct consequence of the crystal nature of space-time: an anisotropy at short wavelengths. Observations of the primordial asymmetry in the cosmic microwave background (G. F. Smoot *et al.*) seem to hint at a slight enhancement of correlations under an angle somewhat larger than 90°, that could easily be the tetrahedral angle $\arccos(-1/3)$.

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